The Energetic Implications of Using Deforming Reference Descriptions to Simulate the Motion of Incompressible, Newtonian Fluids

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Abstract

In this work the issue of whether key energetic properties (nonlinear, exponential–type dissipation in the absence of forcing and long–term stability under conditions of time dependent loading) are automatically inherited by deforming reference descriptions is resolved. These properties are intrinsic to real flows and the conventional Navier–Stokes equations. A completely general reference description of an incompressible, Newtonian fluid, which reconciles the differences between opposing schools of thought in the literature is derived for the purposes of this investigation.

The work subsequently focusses on establishing a class of time discretisations which inherit these self–same energetic properties, irrespective of the time increment employed. The findings of this analysis have profound consequences for the use of certain classes of finite difference schemes in the context of deforming references. It is significant that many algorithms presently in use do not automatically inherit the fundamental qualitative features of the dynamics. An "updated" approach as a means of avoiding ever burgeoning deformation gradients and a still further simplified implementation are further topics explored.

Keywords: Energy Conservation; Incompressible, Newtonian Fluid; Completely General Reference Description; Arbitrary Lagrangian Eulerian; A.L.E.; Rigid Body in a Fluid; Free Surface; Finite Elements; New Poincaré Inequality.

1 Introduction

Descriptions of fluid motion are conventionally based on the principles of conservation of mass and linear momentum. One might hope that all such descriptions would accordingly exhibit key energetic properties (nonlinear, exponential–type dissipation in the absence

of forcing and long–term stability under conditions of time dependent loading) consistant with the principle of energy conservation. These properties are intrinsic to real flows and the conventional, Eulerian Navier–Stokes equations.

A completely general reference description of an incompressible, Newtonian fluid, which reconciles the differences between the so–called arbitrary Lagrangian Eulerian (A.L.E.) formulation of Hughes, Liu and Zimmerman [4] (deformation gradients absent) and that of Soulaimani, Fortin, Dhatt and Ouellet [9] (deformation gradients present, but use is problematic), is derived for the purposes of this investigation. The implications of the resulting description are investigated in the context of energy conservation in a similar, but broader, approach to that taken by others (eg. Simo and Armero [8]) for the conventional, Eulerian Navier–Stokes equations.

The work subsequently focusses on establishing a class of time discretisations which inherit these self–same energetic properties irrespective of the time increment employed. The findings of this analysis have profound consequences for the use of certain classes of difference schemes in the context of deforming references. It is significant that many algorithms presently in use do not automatically inherit the fundamental qualitative features of the dynamics. An "updated" approach as a means of avoiding ever burgeoning deformation gradients which arise from the accumulated step–wise deformation of meshes and a still further simplified implementation are further topics explored.

The main conclusions of this work are based on a new inequality and a number of lemmas. These lemmas are mainly concerned with the new convective term. The new inequality is used in place of where the Poincaré–Friedrichs inequality might otherwise have limited the analysis. This analysis is extended in that non–zero boundaries, so–called free boundaries and time–dependent loads are considered.

2 A Completely General Reference

The implementation of most numerical time integration schemes would be problematic were a conventional Eulerian[1] description of fluid motion to be used in instances involving deforming domains. The reason is that most numerical time integration schemes require successive function evaluation at fixed spatial locations. On the other hand meshes rapidly snarl when purely Lagrangian[2] descriptions are used.

Eulerian and Lagrangian references are just two, specific examples of an unlimited number of configurations over which to define fields used to describe the dynamics of deforming continua. They are both special cases of a more general reference description, a description in which the referential configuration is deformed at will. A deforming finite element mesh would be a good example of just such a deforming reference in practice. The transformation to the completely general reference involves coordinates where used as spatial variables only and the resultant description is therefore inertial in the same

[1] Eulerian or spatial descriptions are in terms of fields defined over the current configuration.

[2] Lagrangian or material descriptions are made in terms of fields defined over a reference (material) configuration.

way as Lagrangian descriptions are.

2.1 Notation

Consider a material body which occupies a domain Ω at time t. The material domain, Ω_0, is that corresponding to time $t = t_0$ (the reference time, t_0, is conventionally, but not always, zero). A third configuration, $\tilde{\Omega}$, which is chosen arbitrarily is also defined for the purposes of this work. The three domains are related in the sense that points in one domain may be obtained as one–to–one invertible maps from points in another.

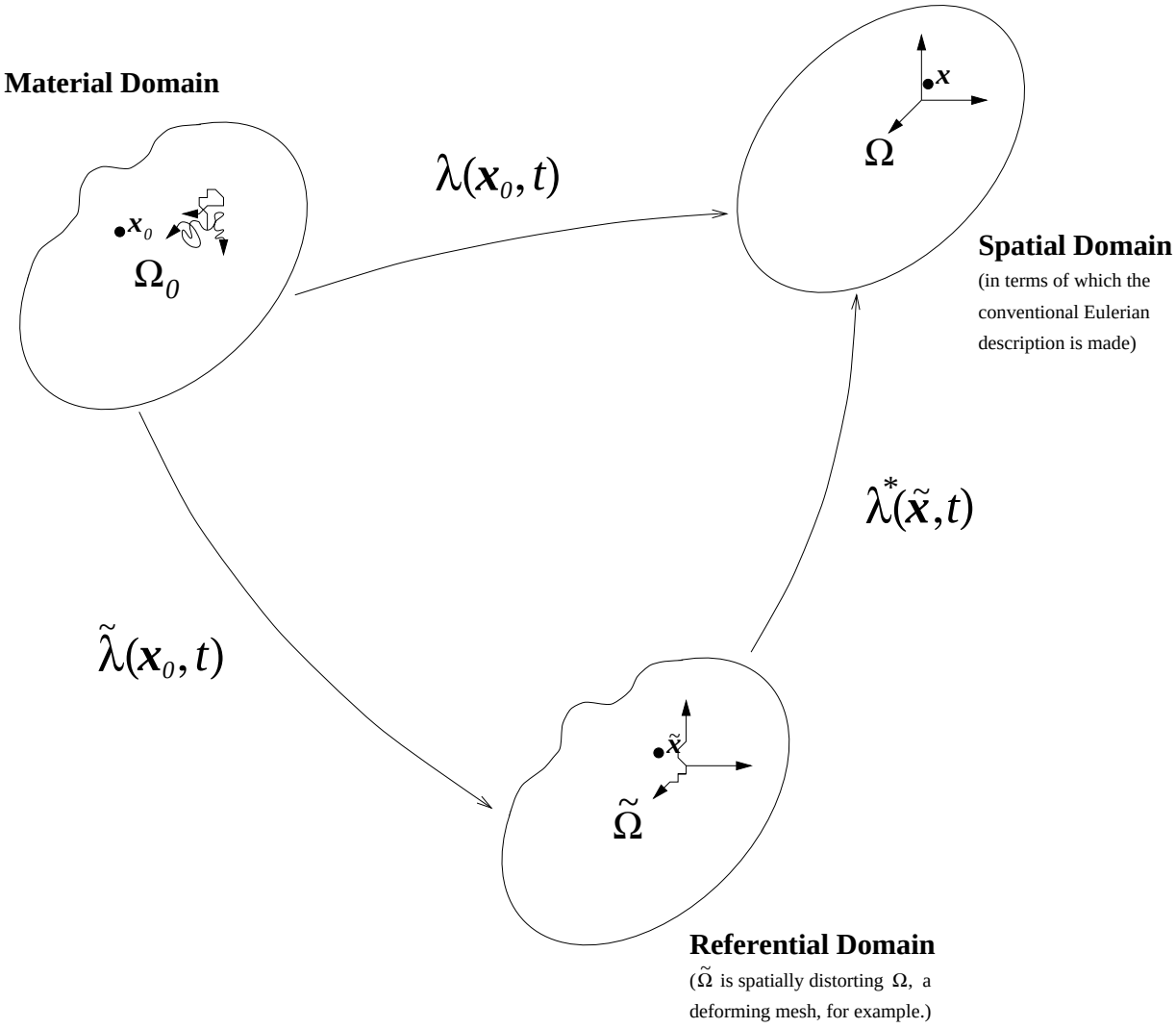

Figure 1: Schematic Diagram of Domains and Mappings Used in a Completely General Reference Description

For any general function $f(\boldsymbol{x}, t)$, a function, $\tilde{f}(\tilde{\boldsymbol{x}}, t) \equiv f(\boldsymbol{\lambda}^*(\tilde{\boldsymbol{x}}, t), t)$, can be defined in terms of the domains and one–to–one, invertible mappings illustrated in Figure 1. Similarly, $f_0(\boldsymbol{x}_0, t) \equiv f(\boldsymbol{\lambda}(\boldsymbol{x}_0, t), t)$ can be defined. This notation can be generalised for

the component–wise definition of higher order tensors. The key to understanding much of this work lies possibly in adopting a component-wise defined notation.

In contrast to the function notation just established, the definition of the operators $\tilde{\nabla}$ and $\widetilde{\text{div}}$ is not based on ∇ and div. They are instead the referential counterparts, that is

$$\tilde{\nabla} = \frac{\partial}{\partial \tilde{\boldsymbol{x}}} \qquad \text{and} \qquad \widetilde{\text{div}} = \frac{\partial}{\partial \tilde{x}_1} + \frac{\partial}{\partial \tilde{x}_2} + \frac{\partial}{\partial \tilde{x}_3}.$$

The notation $\boldsymbol{A} : \boldsymbol{B}$ is used to denote the matrix inner product $A_{ij}B_{ij}$ throughout this work, $\langle \cdot\,,\, \cdot \rangle_{L^2(\,\cdot\,)}$ denotes the L^2 inner product and $||\cdot||_{L^2(\,\cdot\,)}$ the L^2 norm.

2.2 Some General Results for Functions Defined on the Three Domains

Three important results are necessary for the derivation of the completely general reference description and these are presented below.

The Material Derivative in Terms of a Completely General Reference

The material derivative of any vector field $\tilde{\boldsymbol{v}}$ in terms of a completely general, reference is

$$\frac{\partial \tilde{\boldsymbol{v}}}{\partial t} + \tilde{\nabla}\tilde{\boldsymbol{v}} \left[\tilde{\boldsymbol{F}}^{-1} (\tilde{\boldsymbol{v}} - \tilde{\boldsymbol{v}}^{ref}) \right]. \tag{1}$$

where $\tilde{\boldsymbol{v}}^{ref}$ is the velocity of the reference deformation, and $\tilde{\boldsymbol{F}}$ is the deformation gradient given by

$$\tilde{\boldsymbol{F}}(\tilde{\boldsymbol{x}}) = \frac{\partial \boldsymbol{\lambda}^*}{\partial \tilde{\boldsymbol{x}}}.$$

This result (taken from Hughes, Liu and Zimmerman [4]) is obtained by recalling that the material derivative (total derivative) is the derivative with respect to time in the material configuration. Thus

$$\begin{aligned} \frac{D\tilde{v}_i}{Dt} &= \frac{\partial}{\partial t}\{\tilde{v}_i(\tilde{\boldsymbol{\lambda}}(\boldsymbol{x}_0, t), t)\} \\ &= \frac{\partial \tilde{v}_i}{\partial t} + \frac{\partial \tilde{v}_i}{\partial \tilde{x}_j}\frac{\partial \tilde{\lambda}_j}{\partial t}\ . \end{aligned} \tag{2}$$

A more practical expression is needed for $\dfrac{\partial \tilde{\lambda}_j}{\partial t}$ (the velocity as perceived in the distorting reference). This can be obtained by considering

$$\lambda_k(\boldsymbol{x}_0, t) = \lambda_k^*(\tilde{\boldsymbol{\lambda}}(\boldsymbol{x}_0, t), t) \qquad \text{(see Figure 1 on page 3)}$$

so that

$$\left.\frac{\partial \lambda_k}{\partial t}\right|_{\boldsymbol{x}_0 \ fixed} = \left.\frac{\partial \lambda_k^*}{\partial t}\right|_{\tilde{\boldsymbol{x}} \ fixed} + \frac{\partial \lambda_k^*}{\partial \tilde{x}_j}\frac{\partial \tilde{\lambda}_j}{\partial t}$$

or

$$\frac{\partial \tilde{\lambda}_j}{\partial t} = \frac{\partial \tilde{x}_j}{\partial x_k} \left(\left. \frac{\partial \lambda_k}{\partial t} \right|_{\boldsymbol{x}_0 \ fixed} - \left. \frac{\partial \lambda_k^*}{\partial t} \right|_{\tilde{\boldsymbol{x}} \ fixed} \right).$$

Substituting this expression into equation (2), the desired, suitably practicable result is obtained.

An Element of Area in Terms of a Distorting Reference

The second important result can be recalled from general continuum mechanics. Consider an element of area, size dA, with an outward unit normal $\boldsymbol{n}$. Then

$$\boldsymbol{n} dA = \tilde{\boldsymbol{F}}^{-t} \tilde{\boldsymbol{N}} \tilde{J} d\tilde{A} \tag{3}$$

where $d\tilde{A}$ and $\tilde{\boldsymbol{N}}$ denote the respective analogous size and outward unit normal of this element of area in the referential configuration (capital "n" so as to remain consistent with the notation, since $\tilde{N}_i \neq n_i$ in this case) and $\tilde{J} = \det \tilde{\boldsymbol{F}}$. This result is demonstrated in most popular textbooks on continuum mechanics (eg. Lai, Rubin and Krempl [6]).

The Kinematic Result $\dot{\mathcal{J}}_0 = \mathcal{J}_0 \operatorname{div} \boldsymbol{v}$

The material derivative of the Jacobian $\mathcal{J}_0$ is given by the relation

$$\dot{\mathcal{J}}_0 = \mathcal{J}_0 \operatorname{div} \boldsymbol{v}$$

where $\mathcal{J}_0$ is defined as follows,

$$\mathcal{J}_0 \equiv \det \left\{ \frac{\partial \boldsymbol{\lambda}}{\partial \boldsymbol{x}_0} \right\}.$$

This result is demonstrated in most popular textbooks on continuum mechanics (eg. Lai, Rubin and Krempl [6]).

2.3 Derivation of the Completely General Equation

One way in which to derive a completely general reference description of an incompressible, Newtonian fluid is to start with the balance laws in global (integral) form, and to make the necessary substitutions in these integrals. The desired numerical implementation (similar to the conventional Navier–Stokes one which has been thoroughly investigated and found to be stable) is then obtained.

Conservation of Mass

Let $\Omega(t)$ be an arbitrary sub–volume of material. The principle of conservation of mass states that

$$\frac{d}{dt} \int_{\Omega(t)} \rho d\Omega \ = \ 0 \qquad \textit{(rate of change of mass with time = 0)}$$

$$
\begin{aligned}
\frac{d}{dt}\int_{\Omega_0}\rho_0\mathcal{J}_0 d\Omega_0 &= 0 && \textit{(reformulating in terms of the material configuration, } \Omega_0\textit{.)}\\
\int_{\Omega_0}\frac{\partial}{\partial t}\left\{\rho_0\mathcal{J}_0\right\} d\Omega_0 &= 0 && \textit{(since limits are not time dependent in the material configuration.)}\\
\int_{\Omega_0}\left(\rho_0\dot{\mathcal{J}}_0 \;+\; \dot{\rho}_0\mathcal{J}_0\right) d\Omega_0 &= 0 && \textit{(by the chain rule)}\\
\int_{\Omega(t)}\left(\dot{\rho} \;+\; \rho\,\mathrm{div}\,\boldsymbol{v}\right) d\Omega &= 0 && \textit{(using the kinematic result } \dot{\mathcal{J}}_0 = \mathcal{J}_0\mathrm{div}\,\boldsymbol{v}\textit{)}\\
\int_{\tilde{\Omega}(t)}\left(\dot{\rho} \;+\; \rho\frac{\partial\tilde{v}_i}{\partial\tilde{x}_j}\frac{\partial\tilde{x}_j}{\partial x_i}\right)\tilde{J}d\tilde{\Omega} &= 0 && \textit{(reformulating in terms of the distorting referential configuration, } \tilde{\Omega}(t)\textit{.)}\\
\Rightarrow\left(\dot{\rho} \;+\; \rho\tilde{\nabla}\tilde{\boldsymbol{v}} : \tilde{\boldsymbol{F}}^{-t}\right)\tilde{J} &= 0 && \textit{(integrand must be zero since the volume was arbitrary.)}
\end{aligned}
\tag{4}
$$

Thus, for a material of constant, non–zero density,

$$\tilde{\nabla}\tilde{\boldsymbol{v}} : \tilde{\boldsymbol{F}}^{-t} = 0 \qquad \text{since} \qquad \tilde{J} \neq 0 \qquad \textit{(mappings are one-to-one and invertible).}$$

Notice also that equation (4) implies

$$\frac{\partial}{\partial t}\left\{\rho_0\mathcal{J}_0\right\} \;=\; 0 \tag{5}$$

since the volume was arbitrary and the integrand must therefore be zero.

Conservation of Linear Momentum (and Mass)

The principle of conservation of linear momentum for an arbitrary volume of material $\Omega(t)$ with boundary $\Gamma(t)$ states that

$$\frac{d}{dt}\int_{\Omega(t)}\rho\boldsymbol{v}d\Omega = \int_{\Omega(t)}\rho\boldsymbol{b}d\Omega \;+\; \int_{\Gamma(t)}\boldsymbol{\sigma}\boldsymbol{n}dA \tag{6}$$

where ρ is density, $\boldsymbol{b}$ is the body force per unit mass, $\boldsymbol{\sigma}$ is the stress, $\boldsymbol{n}$ the outward unit normal to the boundary and $\boldsymbol{v}$ is the velocity. The term on the lefthand side can be rewritten as follows:

$$
\begin{aligned}
\frac{d}{dt}\int_{\Omega(t)}\rho\boldsymbol{v}d\Omega &= \frac{d}{dt}\int_{\Omega_0}\rho_0\boldsymbol{v}_0\mathcal{J}_0 d\Omega_0 && \textit{(Reformulating in terms of the material configuration, } \Omega_0\textit{.)}\\
&= \int_{\Omega_0}\frac{\partial}{\partial t}\left\{\rho_0\boldsymbol{v}_0\mathcal{J}_0\right\} d\Omega_0 && \textit{(Since limits are not time dependent in the material configuration.)}
\end{aligned}
$$

$$= \int_{\Omega_0} \left(\frac{\partial \boldsymbol{v}_0}{\partial t} \rho_0 \mathcal{J}_0 \; + \; \boldsymbol{v}_0 \frac{\partial}{\partial t} \{\rho_0 \mathcal{J}_0\} \right) d\Omega_0$$

$$= \int_{\Omega(t)} \rho \dot{\boldsymbol{v}} d\Omega \qquad \textit{(The second term above is zero as a consequence of equation (5).)}$$

$$= \int_{\tilde{\Omega}(t)} \rho \dot{\tilde{\boldsymbol{v}}} \tilde{J} d\tilde{\Omega} \qquad \textit{(Reformulating in terms of the distorting referential configuration, } \tilde{\Omega}.\textit{)}$$

$$= \int_{\tilde{\Omega}(t)} \rho \left(\frac{\partial \tilde{\boldsymbol{v}}}{\partial t} + \tilde{\nabla} \tilde{\boldsymbol{v}} \left[\tilde{\boldsymbol{F}}^{-1} (\tilde{\boldsymbol{v}} - \tilde{\boldsymbol{v}}^{ref}) \right] \right) \tilde{J} d\tilde{\Omega} \qquad \textit{(Using result (1) on page 4)}$$

where $\dot{\boldsymbol{v}}$ denotes the material derivative of $\boldsymbol{v}$. The surface integral becomes

$$\int_{\Gamma(t)} \boldsymbol{\sigma} \boldsymbol{n} dA = \int_{\tilde{\Gamma}(t)} \tilde{\boldsymbol{\sigma}} \tilde{\boldsymbol{F}}^{-t} \tilde{\boldsymbol{N}} \tilde{J} d\tilde{A} \qquad \textit{(Reformulating in terms of a distorting reference using result (3) on page 5.)}$$

$$= \int_{\tilde{\Omega}(t)} \widetilde{\operatorname{div}} \{\tilde{\boldsymbol{\sigma}} \tilde{\boldsymbol{F}}^{-t} \tilde{J}\} d\tilde{\Omega} \qquad \textit{(By the divergence theorem).}$$

Finally, the term involving body force becomes

$$\int_{\Omega(t)} \rho \boldsymbol{b} d\Omega = \int_{\tilde{\Omega}(t)} \rho \tilde{\boldsymbol{b}} \tilde{J} d\tilde{\Omega} \qquad \textit{(Reformulating in terms of a distorting reference.).}$$

Substituting these expressions into (6), remembering that the volume used in the argument was arbitrary and that the entire integrand must therefore be zero, the conservation principles of linear momentum and mass may be written in primitive form as

$$\rho \left(\frac{\partial \tilde{\boldsymbol{v}}}{\partial t} + \tilde{\nabla} \tilde{\boldsymbol{v}} \tilde{\boldsymbol{F}}^{-1} (\tilde{\boldsymbol{v}} - \tilde{\boldsymbol{v}}^{ref}) \right) \tilde{J} = \rho \tilde{\boldsymbol{b}} \tilde{J} + \widetilde{\operatorname{div}} \, \tilde{\boldsymbol{P}} \tag{7}$$

and

$$\tilde{\nabla} \tilde{\boldsymbol{v}} : \tilde{\boldsymbol{F}}^{-t} = 0 \tag{8}$$

where $\tilde{\boldsymbol{P}}$ is the Piola–Kirchoff stress tensor of the first kind, $\tilde{\boldsymbol{P}} = \tilde{\boldsymbol{\sigma}} \tilde{\boldsymbol{F}}^{-t} \tilde{J}$. In terms of the constitutive relation, $\boldsymbol{\sigma} = -p\boldsymbol{I} + 2\mu \boldsymbol{D}$, for a Newtonian fluid,

$$\tilde{\boldsymbol{P}} = \left(-p\boldsymbol{I} + \mu \left[\tilde{\nabla} \tilde{\boldsymbol{v}} \tilde{\boldsymbol{F}}^{-1} + \left(\tilde{\nabla} \tilde{\boldsymbol{v}} \tilde{\boldsymbol{F}}^{-1} \right)^t \right] \right) \tilde{\boldsymbol{F}}^{-t} \tilde{J} \qquad \text{since} \qquad \tilde{\boldsymbol{D}} = \frac{1}{2} \left(\widetilde{\nabla \boldsymbol{v}} + \left(\widetilde{\nabla \boldsymbol{v}} \right)^t \right).$$

The derivation of a variational formulation is along similar lines as that for the Navier-Stokes equations (the purely Eulerian description). For a fluid of constant density, the variational formulation

$$\rho \int_{\tilde{\Omega}} \tilde{\boldsymbol{w}} \cdot \frac{\partial \tilde{\boldsymbol{v}}}{\partial t} \tilde{J} d\tilde{\Omega} \; + \; \rho \int_{\tilde{\Omega}} \tilde{\boldsymbol{w}} \cdot \tilde{\nabla} \tilde{\boldsymbol{v}} \left[\tilde{\boldsymbol{F}}^{-1} (\tilde{\boldsymbol{v}} - \tilde{\boldsymbol{v}}^{ref}) \right] \tilde{J} d\tilde{\Omega} =$$

$$\rho \int_{\tilde{\Omega}} \tilde{\boldsymbol{w}} \cdot \tilde{\boldsymbol{b}} \tilde{J} d\tilde{\Omega} \; + \; \int_{\tilde{\Omega}} \tilde{p} \tilde{\nabla} \tilde{\boldsymbol{w}} : \tilde{\boldsymbol{F}}^{-t} \tilde{J} d\tilde{\Omega} \; - \; 2\mu \int_{\tilde{\Omega}} \tilde{\boldsymbol{D}}(\tilde{\boldsymbol{w}}) : \tilde{\mathbf{D}}(\tilde{\boldsymbol{v}}) \tilde{J} d\tilde{\Omega}$$

$$+ \rho \int_{\tilde{\Gamma}} \tilde{\boldsymbol{w}} \tilde{\boldsymbol{P}} \tilde{\boldsymbol{N}} d\tilde{\Gamma} \tag{9}$$

$$\int_{\tilde{\Omega}} \tilde{q}\tilde{\nabla}\tilde{\boldsymbol{v}} : \tilde{\boldsymbol{F}}^{-t} d\tilde{\Omega} \;=\; 0 \tag{10}$$

is obtained, where $\tilde{q}$ and $\tilde{\boldsymbol{w}}$ are respectively the arbitrary pressure and velocity of the variational formulation.

2.4 Reconciling the Different Schools of Thought

The equations (7) and (8) are the completely general reference description of an incompressible, Newtonian fluid. They reduce to the so–called A.L.E. equations of Hughes, Liu and Zimmerman [4] for an instant in which spatial and referential configurations coincide. These simplified equations should, however, not be implemented where the implementation requires evaluation about more than one point within each time step (see Section 5 for a further, in–depth explanation). Under such circumstances the equations of Hughes et al. are an arbitrary Lagrangian Eulerian (A.L.E.) description in the very true sense (this is not surprising considering the equations have their origins in the arbitrarily, either Lagrangian or Eulerian programmes of Hirt, Amsden and Cook [3]). This fact is further borne out in observing that key energetic properties, consistant with the principle of energy conservation, are not automatically inherited by the equations of Hughes et. al. in the context of more general references.

The momentum equations of Soulaimani, Fortin, Dhatt and Ouellet [9] are flawed as a result of the mistaken belief that $\tilde{\boldsymbol{\sigma}}\tilde{\boldsymbol{F}}^{-1}\tilde{J}$ is the Piola–Kirchoff stress tensor of the first kind (pg. 268 of Soulaimani et al.). Yet another problem is illustrated by rewriting the conventional incompressibility condition using the chain rule. The new incompressibility condition which arises is most certainly

$$\frac{\partial \tilde{v}_i}{\partial \tilde{x}_j}\frac{\partial \tilde{x}_j}{\partial x_i} = 0 \qquad \text{and not} \qquad \frac{\partial \tilde{v}_i}{\partial \tilde{x}_j}\frac{\partial \tilde{x}_i}{\partial x_j} = 0.$$

Further errors arising (eg. $\hat{J}$ omitted in the first term on the right hand side of the momentum equation, equation (10) on pg. 268 of Soulaimani et al.) make the use of these equations problematic.

3 The Energetic Implications of a Deforming Reference

The effect of quantities parameterising reference deformation on key energetic properties – nonlinear, exponential–type dissipation in the absence of forcing and long–term stability under conditions of time dependent loading – is investigated in this section. These properties,

$$K(\boldsymbol{v}) \leq K(\boldsymbol{v}\mid_{t_0})\, e^{-2\nu C t} \qquad \text{and} \qquad \lim_{t\to\infty} \sup K(\boldsymbol{v}) \leq \frac{M^2}{2\nu^2 C^2}$$

respectively (where $K = \frac{1}{2}\rho\,||\boldsymbol{v}||^2_{L^2(\Omega)}$ is the total kinetic energy), are intrinsic to real flows and the conventional, Eulerian Navier–Stokes equations (see TEMAM [10], [11], CONSTANTIN and FOIAS [1] and SIMO and ARMERO [8] in this regard). The effect of $\tilde{\boldsymbol{v}}^{ref}$ on the afore mentioned aspects of conservation of the quantity

$$\frac{1}{2}\rho\left|\left|\tilde{\boldsymbol{v}}\tilde{J}^{\frac{1}{2}}\right|\right|^2_{L^2(\tilde{\Omega})}$$

is essentially what is being investigated, with a view to establishing a set of conditions under which the discrete approximation can reasonably be expected to inherit these self–same energetic properties.

One might anticipate key energetic properties to be manifest only in instances involving a fixed contributing mass of material, whether its boundaries be dynamic, or not. An analysis of this nature only makes sense in the context of a constant volume of fluid which, for simplicity, will have material limits.

Inequalities of the Poincaré-Friedrichs type are a key feature of any stability analysis of this nature. Gradient containing L^2 terms need to be re–expressed in terms of energy. In the case of a "no slip" ($\boldsymbol{v} = 0$) condition on the entire boundary the situation is straightforward, in that it is possible to use the standard Poincare-Friedrichs inequality: there exists a constant $C_1 > 0$ such that

$$\|\boldsymbol{v}\|_{L^2} \leq C_1\|\nabla\boldsymbol{v}\|_{L^2} \quad \text{for all } \boldsymbol{v} \in [H^1_0(\Omega)]^n.$$

The use of the classical Poincaré–Friedrichs inequality is otherwise identified as a major limitation, even in the conventional Navier–Stokes related analyses. The Poincaré–Friedrichs inequality is only applicable in very limited instances where the value for the entire boundary is stipulated to be identically zero. For boundary conditions of a more general nature, such as those encountered in this study, in which parts of the boundary may be either a free surface or subject to traction conditions, a more suitable inequality is required (notice that subtracting a boundary velocity and analising the resulting equation is not feasible as the equations are nonlinear). The Poincaré–Friedrichs inequality does, furthermore, not hold on subdomains of the domain in question and the constant is not optimal.

Further investigation (COMMUNICATION [7]) reveals a similar result, the so-called Poincaré–Morrey inequality, holds providing the function attains a value of zero somewhere on the boundary. The Poincaré-Morrey inequality states that a constant $C_2 > 0$ exists such that

$$\|\boldsymbol{v}\|_{L^2} \leq C_2\|\nabla\boldsymbol{v}\|_{L^2} \quad \text{for all } \boldsymbol{v} \in [H^1_0(\Omega)]^n.$$

The proof of the Poincaré-Morrey inequality is, however, similar to that of one of Korn's inequalities (see, for example, KIKUCHI and ODEN [5]). In particluar, it is non–constructive, by contradiction and the constant cannot therefore be determined as part of the proof. Viewed in this light the forthcoming inequality amounts to a specification of the hypothetical constant in the Poincaré–Morrey inequality for domains of a particular geometry. The particular types of geometry considered are those that arise in problems involving the motion of rigid bodies such as pebbles on the sea bed; thus a free surface is present, and the domain may be multiply connected.

INEQUALITY 1 (A NEW "POINCARÉ" INEQUALITY) *Suppose* $\boldsymbol{v}$ *is continuous and differentiable to first order and that* $\boldsymbol{v}$ *attains a maximum absolute value,* c*, on an included, finite neighbourhood of minimum radius* $R_{\min}$ *about a point* $\boldsymbol{x}^{\text{origin}}$ *(as depicted in Figure 2).*

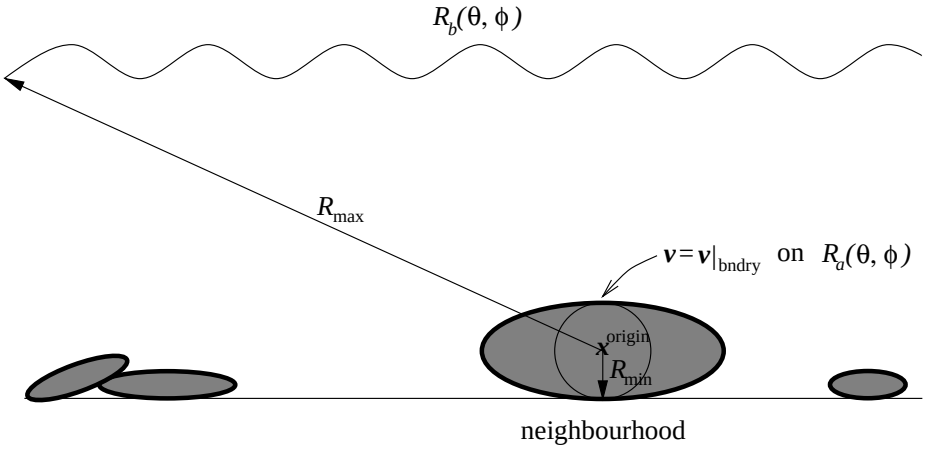

Figure 2: A Finite Neighbourhood of Minimum Radius $R_{\min}$ About a Point $\boldsymbol{x}^{\text{origin}}$.

If Ω *is a bounded, star–shaped (about a point* $\boldsymbol{x}^{\text{origin}}$*)*[1] *domain in* R^3*, then*

$$||\boldsymbol{v}||_{L^2(\Omega)} \leq \left[\frac{(R_{\max}-R_{\min})(R_{\max}^3-R_{\min}^3)}{3R_{\max}R_{\min}}\right]^{\frac{1}{2}} ||\nabla \boldsymbol{v}||_{L^2(\Omega)} + ||c||_{L^2(\Omega)}$$

where $R_{\max}$ *is the distance to the farthest point in* Ω *from* $\boldsymbol{x}^{\text{origin}}$*.*

PROOF: Consider the change to spherical coordinates

$$\breve{v}_i(r,\theta,\phi) = v_i(r\sin\theta\cos\phi - x_1^{\text{origin}}, r\sin\theta\sin\phi - x_2^{\text{origin}}, r\cos\theta - x_3^{\text{origin}})$$

centred on $\boldsymbol{x}^{\text{origin}}$. Suppose the radial limits of the domain and neighbourhood are denoted $R_b(\theta,\phi)$ and $R_a(\theta,\phi)$ respectively. By the fundamental theorem of integral calculus

$$\begin{aligned}
\left(\breve{v}_i(r,\theta,\phi) - \breve{v}_i\,|_{R_a(\theta,\phi)}\right)^2 &= \left(\int_{R_a(\theta,\phi)}^{r} \frac{\partial \breve{v}_i}{\partial r}(\xi,\theta,\phi)d\xi\right)^2 \\
&= \left(\int_{R_a(\theta,\phi)}^{r} \frac{1}{\xi}\xi\frac{\partial \breve{v}_i}{\partial r}(\xi,\theta,\phi)d\xi\right)^2 \\
&\leq \int_{R_a(\theta,\phi)}^{r} \frac{1}{\xi^2}d\xi \int_{R_a(\theta,\phi)}^{r} \left(\frac{\partial \breve{v}_i}{\partial r}(\xi,\theta,\phi)\right)^2 \xi^2 d\xi
\end{aligned}$$

[1] by which is meant that every point in the domain can be reached by a straight line from $\boldsymbol{x}^{\text{origin}}$ that does not pass outside of Ω

$$
\begin{aligned}
& \textit{(by Schwarz inequality)} \\
\leq\ & \int_{R_{\min}}^{R_{\max}} \frac{1}{\xi^2} d\xi \int_{R_a(\theta,\phi)}^{R_b(\theta,\phi)} \left(\frac{\partial \breve{v}_i}{\partial r}(\xi,\theta,\phi) \right)^2 \xi^2 d\xi \qquad \textit{(for } r \in \breve{\Omega}\textit{)} \\
=\ & \frac{(R_{\max}-R_{\min})}{R_{\max}R_{\min}} \int_{R_a(\theta,\phi)}^{R_b(\theta,\phi)} \left(\frac{\partial \breve{v}_i}{\partial r}(\xi,\theta,\phi) \right)^2 \xi^2 d\xi \\
=\ & \frac{(R_{\max}-R_{\min})}{R_{\max}R_{\min}} \breve{V}_i(\theta,\phi)
\end{aligned}
$$

$$
\text{where} \qquad \breve{V}_i(\theta,\phi) = \int_{R_a(\theta,\phi)}^{R_b(\theta,\phi)} \left(\frac{\partial \breve{v}_i}{\partial r}(\xi,\theta,\phi) \right)^2 \xi^2 d\xi .
$$

Integrating this result over that part of $\breve{\Omega}$ outside the neighbourhood (angular extent being $\Theta_a(\phi) \leq \theta \leq \Theta_b(\phi)$ and $\Phi_a \leq \phi \leq \Phi_b$)

$$
\begin{aligned}
& \int_{\Phi_a}^{\Phi_b} \int_{\Theta_a(\phi)}^{\Theta_b(\phi)} \int_{R_a(\theta,\phi)}^{R_b(\theta,\phi)} \left(\breve{v}_i(r,\theta,\phi) - \breve{v}_i \mid_{R_a(\theta,\phi)} \right)^2 r^2 \sin\theta dr d\theta d\phi \\
& \qquad \leq \frac{(R_{\max}-R_{\min})}{R_{\max}R_{\min}} \int_{\Phi_a}^{\Phi_b} \int_{\Theta_a(\phi)}^{\Theta_b(\phi)} \int_{R_a(\theta,\phi)}^{R_b(\theta,\phi)} \breve{V}_i(\theta,\phi) r^2 \sin\theta dr d\theta d\phi \\
& \qquad \leq \frac{(R_{\max}-R_{\min})}{R_{\max}R_{\min}} \int_{\Phi_a}^{\Phi_b} \int_{\Theta_a(\phi)}^{\Theta_b(\phi)} \breve{V}_i(\theta,\phi) \left(\int_{R_{\min}}^{R_{\max}} r^2 dr \right) \sin\theta d\theta d\phi \\
& \qquad \leq \frac{(R_{\max}-R_{\min})(R_{\max}^3-R_{\min}^3)}{3R_{\max}R_{\min}} \int_{\Phi_a}^{\Phi_b} \int_{\Theta_a(\phi)}^{\Theta_b(\phi)} \int_{R_a(\theta,\phi)}^{R_b(\theta,\phi)} \left(\frac{\partial \breve{v}_i}{\partial r} \right)^2 r^2 \sin\theta dr d\theta d\phi \\
& \qquad \leq \frac{(R_{\max}-R_{\min})(R_{\max}^3-R_{\min}^3)}{3R_{\max}R_{\min}} \int_{\Phi_a}^{\Phi_b} \int_{\Theta_a(\phi)}^{\Theta_b(\phi)} \int_{R_a(\theta,\phi)}^{R_b(\theta,\phi)} \left[\left(\frac{\partial \breve{v}_i}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial \breve{v}_i}{\partial \theta} \right)^2 \right. \\
& \qquad\qquad \left. + \frac{1}{r^2 \sin^2\theta} \left(\frac{\partial \breve{v}_i}{\partial \phi} \right)^2 \right] r^2 \sin\theta dr d\theta d\phi \\
& \qquad = \frac{(R_{\max}-R_{\min})(R_{\max}^3-R_{\min}^3)}{3R_{\max}R_{\min}} \int_{\Phi_a}^{\Phi_b} \int_{\Theta_a(\phi)}^{\Theta_b(\phi)} \int_{R_a(\theta,\phi)}^{R_b(\theta,\phi)} (\nabla \breve{v}_i) \cdot (\nabla \breve{v}_i)\, r^2 \sin\theta dr d\theta d\phi .
\end{aligned}
$$

Changing back to the original rectangular coordinates and defining $\boldsymbol{v} \mid_{\text{bndry}}$ to be a radially constant function throughout Ω which takes the values of $\breve{\boldsymbol{v}} \mid_{R_a(\theta,\phi)}$ for $r = R_a(\theta,\phi)$,

$$
\int_{\Omega_*} (v_i(\boldsymbol{x}) - v_i \mid_{\text{bndry}})^2 \, d\Omega \ \leq\ \frac{(R_{\max}-R_{\min})(R_{\max}^3-R_{\min}^3)}{3R_{\max}R_{\min}} \int_{\Omega_*} (\nabla v_i(\boldsymbol{x})) \cdot (\nabla v_i(\boldsymbol{x}))\, d\Omega
$$

where Ω_* is Ω excluding the neighbourhood. Summing over i,

$$
\int_{\Omega_*} (\boldsymbol{v} - \boldsymbol{v} \mid_{\text{bndry}}) \cdot (\boldsymbol{v} - \boldsymbol{v} \mid_{\text{bndry}})\, d\Omega \ \leq\ \frac{(R_{\max}-R_{\min})(R_{\max}^3-R_{\min}^3)}{3R_{\max}R_{\min}} \int_{\Omega_*} (\nabla \boldsymbol{v}) : (\nabla \boldsymbol{v})\, d\Omega .
$$

Making use of either the Cauchy–Schwarz or triangle inequality,

$$
\left(||\boldsymbol{v}||_{L^2(\Omega_*)} - ||\boldsymbol{v} \mid_{\text{bndry}}||_{L^2(\Omega_*)} \right)^2 \ \leq\ \frac{(R_{\max}-R_{\min})(R_{\max}^3-R_{\min}^3)}{3R_{\max}R_{\min}} \, ||\nabla \boldsymbol{v}||^2_{L^2(\Omega_*)} ,
$$

and remembering that sup $|\ \breve{\boldsymbol{v}}\ |_{R_a(\theta,\phi)}| \leq c$,

$$||\boldsymbol{v}||_{L^2(\Omega_*)} \leq \left[\frac{(R_{\max}-R_{\min})(R_{\max}^3-R_{\min}^3)}{3R_{\max}R_{\min}}\right]^{\frac{1}{2}} ||\nabla \boldsymbol{v}||_{L^2(\Omega_*)} + ||c||_{L^2(\Omega_*)}\,.$$

Consider the terms $||\boldsymbol{v}||_{L^2}$ and $||c||_{L^2}$. Comparing these terms under circumstances of sup $|\ \boldsymbol{v}\ | \leq c$ leads to the conclusion that the inequality holds over the neighbourhood and that the inequality is therefore unaffected when the domain of integration is extended to include the neighbourhood. Of course, the radial extension of $\boldsymbol{v}\ |_{\text{bndry}}$ can be used in place of c in instances where inclusion of the neighbourhood is not required.

This inequality is similar to the Poincaré–Friedrichs inequality when $c = 0$, but is extended to a geometrical subclass of domains which have free and partly non-zero boundaries. It has a further advantage in that the constant is an order of magnitude more optimal when used under the "no slip" Poincaré–Friedrichs condition (under such conditions the domain can always be deconstructed into a number of subdomains in which $R_{\min} = \frac{1}{3}R_{\max}$). The Poincaré–Friedrichs inequality is a special case of the above inequality. The necessary lemma (below) follows naturally from the above inequality.

Lemma 1 (Deviatoric Stress Term Energy) *The kinetic energy satisfies the bound* $\frac{C}{\rho}\tilde{K}(\tilde{\boldsymbol{v}}) \leq \left|\left|\tilde{\boldsymbol{D}}(\tilde{\boldsymbol{v}})\tilde{J}^{\frac{1}{2}}\right|\right|^2_{L^2(\tilde{\Omega})}$, *where C is related to the constant in Inequality 1, $C > 0$.*

Proof: If, in particular, $\boldsymbol{v}\ |_{\text{bndry}} = 0$ in Inequality 1,

$$\begin{aligned} ||\boldsymbol{v}||_{L^2(\Omega)} &\leq \frac{||\nabla \boldsymbol{v}||_{L^2(\Omega)}}{\sqrt{C}} \\ C\frac{1}{2}\,||\boldsymbol{v}||^2_{L^2(\Omega)} &\leq ||\boldsymbol{D}(\boldsymbol{v})||^2_{L^2(\Omega)} \end{aligned}$$

(The relationship between $\boldsymbol{D}$ and $\nabla\boldsymbol{v}$ arises in the context of the original equations involving div $\boldsymbol{\sigma}$. It is because

$$\begin{aligned} D_{ij,j} &= \tfrac{1}{2}\,(v_{i,jj} + v_{j,ij}) \\ &= \tfrac{1}{2}\,(v_{i,jj} + v_{j,ji}) && \textit{(changing the order of differentiation)} \\ &= \tfrac{1}{2}v_{i,jj} && (\text{div}\boldsymbol{v} = 0 \textit{ by incompressibility}), \end{aligned}$$

assuming, of course, that $\boldsymbol{v}$ is continuous and differentiable to first order.) Rewriting in terms of $\tilde{\Omega}$

$$\frac{C}{\rho}\tilde{K}(\tilde{\boldsymbol{v}}) \leq \left|\left|\tilde{\boldsymbol{D}}(\tilde{\boldsymbol{v}})\tilde{J}^{\frac{1}{2}}\right|\right|^2_{L^2(\tilde{\Omega})}\,.$$

The following lemma is vital to the deforming reference analysis in particular. It will form the basis to the next lemma and another (on page 18) concerned with the time discrete analysis.

LEMMA 2 (BASIC TO LEMMAS 3 AND 5) *The relation*

$$\left\langle \tilde{\boldsymbol{u}}, (\tilde{\nabla}\tilde{\boldsymbol{v}})\tilde{\boldsymbol{F}}^{-1}\tilde{\boldsymbol{w}}\tilde{J}\right\rangle_{L^2(\tilde{\Omega})} = -\left\langle \tilde{\boldsymbol{v}}, (\tilde{\nabla}\tilde{\boldsymbol{u}})\tilde{\boldsymbol{F}}^{-1}\tilde{\boldsymbol{w}}\tilde{J}\right\rangle_{L^2(\tilde{\Omega})} - \left\langle \tilde{\boldsymbol{u}}\left(\tilde{\nabla}\tilde{\boldsymbol{w}} : \tilde{\boldsymbol{F}}^{-t}\right), \tilde{\boldsymbol{v}}\tilde{J}\right\rangle_{L^2(\tilde{\Omega})}$$

is valid for

$$\tilde{\boldsymbol{w}} \in W = \left\{\tilde{\boldsymbol{w}} \ : \ \tilde{\boldsymbol{w}} = \boldsymbol{0} \text{ or } \tilde{\boldsymbol{F}}^{-t}\tilde{\boldsymbol{N}}\cdot\tilde{\boldsymbol{w}} = 0 \text{ on } \tilde{\Gamma}\right\}.$$

PROOF: Consider $\tilde{\boldsymbol{u}}\cdot(\tilde{\nabla}\tilde{\boldsymbol{v}})\tilde{\boldsymbol{F}}^{-1}\tilde{\boldsymbol{w}}\tilde{J}$:

$$\tilde{u}_i\tilde{v}_{i,j}\tilde{F}^{-1}_{jk}\tilde{w}_k\tilde{J} \ = \ -\tilde{u}_{i,j}\tilde{v}_i\tilde{F}^{-1}_{jk}\tilde{w}_k\tilde{J} - \tilde{u}_i\tilde{v}_i(\tilde{F}^{-1}_{jk}\tilde{w}_k\tilde{J})_{,j} + (\tilde{u}_i\tilde{v}_i\tilde{F}^{-1}_{jk}\tilde{w}_k\tilde{J})_{,j}$$

by the product rule. In the terms arising from $(\tilde{F}^{-1}_{jk}\tilde{w}_k\tilde{J})_{,j}$, both $\tilde{F}^{-1}_{jk,j}$ and $\tilde{J}_{,j}\tilde{F}^{-1}_{jk}$ vanish under the condtions specified (in section 2.4) for equations of HUGHES, LIU and ZIMMERMAN [4] to be a completely general reference description. Thus

$$\tilde{u}_i\tilde{v}_{i,j}\tilde{F}^{-1}_{jk}\tilde{w}_k\tilde{J} \ = \ -\tilde{u}_{i,j}\tilde{v}_i\tilde{F}^{-1}_{jk}\tilde{w}_k\tilde{J} - \tilde{u}_i\tilde{v}_i\tilde{F}^{-1}_{jk}\tilde{w}_{k,j}\tilde{J} + (\tilde{u}_i\tilde{v}_i\tilde{F}^{-1}_{jk}\tilde{w}_k\tilde{J})_{,j}.$$

Integrating over the domain $\tilde{\Omega}$ and applying the divergence theorem,

$$\begin{aligned}\left\langle \tilde{\boldsymbol{u}}, (\tilde{\nabla}\tilde{\boldsymbol{v}})\tilde{\boldsymbol{F}}^{-1}\tilde{\boldsymbol{w}}\tilde{J}\right\rangle_{L^2(\tilde{\Omega})} \ = \ & -\left\langle \tilde{\boldsymbol{v}}, (\tilde{\nabla}\tilde{\boldsymbol{u}})\tilde{\boldsymbol{F}}^{-1}\tilde{\boldsymbol{w}}\tilde{J}\right\rangle_{L^2(\tilde{\Omega})} \\ & -\left\langle \tilde{\boldsymbol{u}}\left(\tilde{\nabla}\tilde{\boldsymbol{w}} : \tilde{\boldsymbol{F}}^{-t}\right), \tilde{\boldsymbol{v}}\tilde{J}\right\rangle_{L^2(\tilde{\Omega})} \\ & +\left\langle \tilde{\boldsymbol{u}}, \tilde{\boldsymbol{v}}\left(\tilde{\boldsymbol{F}}^{-t}\tilde{\boldsymbol{N}}\cdot\tilde{\boldsymbol{w}}\right)\tilde{J}\right\rangle_{L^2(\tilde{\Gamma})} \end{aligned} \tag{11}$$

The condition of this lemma dictates the manner in which the reference must deform to ensure that the equation will inherit the desired energetic properties. This lemma is crucial to the deforming reference analysis. The lemma immediately below will facilitate the elimination of the convective energy rate in the forthcoming analysis.

LEMMA 3 (CONVECTIVE ENERGY RATE) *The relation*

$$-\rho\left\langle \tilde{\boldsymbol{v}}, (\tilde{\nabla}\tilde{\boldsymbol{v}})\tilde{\boldsymbol{F}}^{-1}\left(\tilde{\boldsymbol{v}} - \tilde{\boldsymbol{v}}^{ref}\right)\tilde{J}\right\rangle_{L^2(\tilde{\Omega})} \ = \ -\frac{1}{2}\rho\left\langle \tilde{\boldsymbol{v}}, \tilde{\boldsymbol{v}}\frac{\partial\tilde{J}}{\partial t}\right\rangle_{L^2(\tilde{\Omega})}$$

is valid in instances where a purely Lagrangian description is used to track free boundaries and/or boundaries are of a fixed impermeable type.

PROOF: In instances where a purely Lagrangian description is used to track free boundaries, $\tilde{\boldsymbol{v}} - \tilde{\boldsymbol{v}}^{ref}$ vanishes, as does $\tilde{\boldsymbol{F}}^{-t}\tilde{\boldsymbol{N}}\cdot\tilde{\boldsymbol{v}}$ at fixed impermeable boundaries. The condition at the boundary for Lemma 2 is therefore satisfied. Thus the term

$$\begin{aligned} -\rho\left\langle \tilde{\boldsymbol{v}}, (\tilde{\nabla}\tilde{\boldsymbol{v}})\tilde{\boldsymbol{F}}^{-1}\left(\tilde{\boldsymbol{v}} - \tilde{\boldsymbol{v}}^{ref}\right)\tilde{J}\right\rangle_{L^2(\tilde{\Omega})} \ = \ & \rho\left\langle \tilde{\boldsymbol{v}}, (\tilde{\nabla}\tilde{\boldsymbol{v}})\tilde{\boldsymbol{F}}^{-1}\left(\tilde{\boldsymbol{v}} - \tilde{\boldsymbol{v}}^{ref}\right)\tilde{J}\right\rangle_{L^2(\tilde{\Omega})} \\ & +\rho\left\langle \tilde{\boldsymbol{v}}\left(\tilde{\nabla}\left(\tilde{\boldsymbol{v}} - \tilde{\boldsymbol{v}}^{ref}\right) : \tilde{\boldsymbol{F}}^{-t}\right), \tilde{\boldsymbol{v}}\tilde{J}\right\rangle_{L^2(\tilde{\Omega})} \end{aligned}$$

$$
\begin{aligned}
&= \rho\left\langle \tilde{\boldsymbol{v}}, (\tilde{\nabla}\tilde{\boldsymbol{v}})\tilde{\boldsymbol{F}}^{-1}\left(\tilde{\boldsymbol{v}}-\tilde{\boldsymbol{v}}^{ref}\right)\tilde{J}\right\rangle_{L^2(\tilde{\Omega})} \\
&\quad -\rho\left\langle \tilde{\boldsymbol{v}}\left(\tilde{\nabla}\tilde{\boldsymbol{v}}^{ref} : \tilde{\boldsymbol{F}}^{-t}\right), \tilde{\boldsymbol{v}}\tilde{J}\right\rangle_{L^2(\tilde{\Omega})} \quad \textit{(by incomp–ressibility)} \\
&= -\frac{1}{2}\rho\left\langle \tilde{\boldsymbol{v}}\left(\tilde{\nabla}\tilde{\boldsymbol{v}}^{ref} : \tilde{\boldsymbol{F}}^{-t}\right), \tilde{\boldsymbol{v}}\tilde{J}\right\rangle_{L^2(\tilde{\Omega})} \\
&= -\frac{1}{2}\rho\left\langle \tilde{\boldsymbol{v}}, \tilde{\boldsymbol{v}}\frac{\partial \tilde{J}}{\partial t}\right\rangle_{L^2(\tilde{\Omega})}
\end{aligned}
$$

since $\dfrac{\partial \tilde{J}}{\partial t} = \tilde{J} div \boldsymbol{v}^{ref}$ (which is $\tilde{J}\tilde{\nabla}\tilde{\boldsymbol{v}}^{ref} : \tilde{\boldsymbol{F}}^{-t}$) in the same vein as $\dot{\mathcal{J}}_0 = \mathcal{J}_0 \operatorname{div} \boldsymbol{v}$ (the kinematic result on used earlier).

This lemma concludes the preliminaries required for the deforming reference energy analysis.

3.1 Exponential Dissipation in the Absence of Forcing

The issue of whether nonlinear, exponential–type dissipation in the absence of forcing is a property intrinsic to the deforming reference description is resolved as follows.

THEOREM 1 (EXPONENTIAL DISSIPATION IN THE ABSENCE OF FORCING) *A sufficient condition for the completely general reference description to inherit nonlinear, exponential type energy dissipation*

$$\tilde{K}(\tilde{\boldsymbol{v}}) \leq \tilde{K}(\tilde{\boldsymbol{v}}\mid_{t_0})\; e^{-2\nu C t}$$

(where $\tilde{K}(\tilde{\boldsymbol{v}}) \equiv \dfrac{1}{2}\rho\left\|\tilde{\boldsymbol{v}}\tilde{J}^{\frac{1}{2}}\right\|^2_{L^2(\tilde{\Omega})}$*) in the absence of forcing (an intrinsic feature of real flows and the conventional, Eulerian Navier–Stokes equations) is that the reference moves in a purely Lagrangian fashion at free boundaries.*

PROOF: The first step towards formulating an expression involving the kinetic energy is to substitute $\tilde{\boldsymbol{v}}$ for $\tilde{\boldsymbol{w}}$ in the variational momentum equation (9) on page 7. Then

$$
\begin{aligned}
\rho\left\langle \tilde{\boldsymbol{v}}, \frac{\partial \tilde{\boldsymbol{v}}}{\partial t}\tilde{J}\right\rangle_{L^2(\tilde{\Omega})} &= \left\langle \tilde{p}\tilde{\nabla}\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{F}}^{-t}\tilde{J}\right\rangle_{L^2(\tilde{\Omega})} - 2\mu\left\langle \tilde{\boldsymbol{D}}(\tilde{\boldsymbol{v}}), \tilde{\boldsymbol{D}}(\tilde{\boldsymbol{v}})\tilde{J}\right\rangle_{L^2(\tilde{\Omega})} \\
&\quad -\rho\left\langle \tilde{\boldsymbol{v}}, (\tilde{\nabla}\tilde{\boldsymbol{v}})\tilde{\boldsymbol{F}}^{-1}\left(\tilde{\boldsymbol{v}}-\tilde{\boldsymbol{v}}^{ref}\right)\tilde{J}\right\rangle_{L^2(\tilde{\Omega})} \\
&\quad +\rho\left\langle \tilde{\boldsymbol{v}}, \tilde{\boldsymbol{b}}\tilde{J}\right\rangle_{L^2(\tilde{\Omega})} + \left\langle \tilde{\boldsymbol{v}}, \tilde{\boldsymbol{P}}\tilde{\boldsymbol{N}}\right\rangle_{L^2(\tilde{\Gamma})}. \qquad (12)
\end{aligned}
$$

The term containing the pressure, that is

$$\left\langle \tilde{p}\tilde{\nabla}\tilde{\boldsymbol{v}} : \tilde{\boldsymbol{F}}^{-t}\tilde{J}\right\rangle_{L^2(\tilde{\Omega})},$$

vanishes as a result of incompressibility (equation (8)). The order of integration and differentiation are interchangeable (limits are time–independent in the reference which tracks the free boundary perfectly – a description which becomes fully Lagrangian at boundaries was stipulated). Equation (12) can be rewritten

$$\begin{aligned}\frac{1}{2}\rho\left(\frac{d}{dt}\left|\left|\tilde{\boldsymbol{v}}\tilde{J}^{\frac{1}{2}}\right|\right|^2_{L^2(\tilde{\Omega})}-\left\langle\tilde{\boldsymbol{v}},\tilde{\boldsymbol{v}}\frac{\partial\tilde{J}}{\partial t}\right\rangle_{L^2(\tilde{\Omega})}\right) &= -2\mu\left|\left|\tilde{\boldsymbol{D}}\tilde{J}^{\frac{1}{2}}\right|\right|^2_{L^2(\tilde{\Omega})}\\ &\quad -\rho\left\langle\tilde{\boldsymbol{v}},(\tilde{\nabla}\tilde{\boldsymbol{v}})\tilde{\boldsymbol{F}}^{-1}\left(\tilde{\boldsymbol{v}}-\tilde{\boldsymbol{v}}^{ref}\right)\tilde{J}\right\rangle_{L^2(\tilde{\Omega})}\\ &\quad +\rho\left\langle\tilde{\boldsymbol{v}},\tilde{\boldsymbol{b}}\tilde{J}\right\rangle_{L^2(\tilde{\Omega})}+\left\langle\tilde{\boldsymbol{v}},\tilde{\boldsymbol{P}}\tilde{\boldsymbol{N}}\right\rangle_{L^2(\tilde{\Gamma})}\end{aligned}$$

as a result. The conditions of Lemma 3 are also satisfied for a description which becomes fully Lagrangian at free boundaries and an expression

$$\frac{d\tilde{K}(\tilde{\boldsymbol{v}})}{dt} = -2\mu\left|\left|\tilde{\boldsymbol{D}}\tilde{J}^{\frac{1}{2}}\right|\right|^2_{L^2(\tilde{\Omega})}+\rho\left\langle\tilde{\boldsymbol{v}},\tilde{\boldsymbol{b}}\tilde{J}\right\rangle_{L^2(\tilde{\Omega})}+\left\langle\tilde{\boldsymbol{v}},\tilde{\boldsymbol{P}}\tilde{\boldsymbol{N}}\right\rangle_{L^2(\tilde{\Gamma})}$$

is therefore obtained, where $\tilde{K}=\frac{1}{2}\rho\left|\left|\tilde{\boldsymbol{v}}\tilde{J}^{\frac{1}{2}}\right|\right|^2_{L^2(\tilde{\Omega})}$ is the total kinetic energy. Using Lemma 1

$$\frac{d\tilde{K}(\tilde{\boldsymbol{v}})}{dt} \leq -2\nu C\tilde{K}(\tilde{\boldsymbol{v}})+\rho\left\langle\tilde{\boldsymbol{v}},\tilde{\boldsymbol{b}}\tilde{J}\right\rangle_{L^2(\tilde{\Omega})}+\left\langle\tilde{\boldsymbol{v}},\tilde{\boldsymbol{P}}\tilde{\boldsymbol{N}}\right\rangle_{L^2(\tilde{\Gamma})}. \tag{13}$$

Equation (13) has a solution of the form

$$\tilde{K}\leq\tilde{K}(\tilde{\boldsymbol{v}}\mid_{t_0})\,e^{-2\nu Ct}$$

in the absence of forcing (*"no forcing"* $\Rightarrow\tilde{\boldsymbol{b}}=\tilde{\boldsymbol{P}}\tilde{\boldsymbol{N}}=\boldsymbol{0}$), providing a purely Lagrangian description is used at free boundaries.

A nonlinear, exponential–type energy dissipation in the absence of forcing is therefore an intrinsic property of the completely general reference description. This contractive flow property is also an intrinsic property of the conventional Navier–Stokes equations.

3.2 Long–Term Stability under Conditions of Time–Dependent Loading

The formulation of suitable load and free surface bounds is necessary before the issue of long-term stability (L^2–stability) under conditions of time–dependent loading can be resolved. The following lemma facilitates the formulation of load and free surface bounds.

Lemma 4 (Force, Free Surface Bounds) *The inequality*

$$\begin{aligned}\rho\left\langle\tilde{\boldsymbol{v}},\tilde{\boldsymbol{b}}\tilde{J}\right\rangle_{L^2(\tilde{\Omega})}+\left\langle\tilde{\boldsymbol{v}},\tilde{\boldsymbol{P}}\tilde{\boldsymbol{N}}\right\rangle_{L^2(\tilde{\Gamma})} &\leq \frac{\nu C}{2}\left(\rho\left|\left|\tilde{\boldsymbol{v}}\tilde{J}^{\frac{1}{2}}\right|\right|^2_{L^2(\tilde{\Omega})}+||\tilde{\boldsymbol{v}}||^2_{L^2(\tilde{\Gamma})}\right)\\ &\quad +\frac{1}{2\nu C}\left(\rho\left|\left|\tilde{\boldsymbol{b}}\tilde{J}^{\frac{1}{2}}\right|\right|^2_{L^2(\tilde{\Omega})}+\left|\left|\tilde{\boldsymbol{P}}\tilde{\boldsymbol{N}}\right|\right|^2_{L^2(\tilde{\Gamma})}\right)\end{aligned}$$

holds where νC is a constant, $\nu C>0$.

PROOF: In terms of the Cauchy–Schwarz inequality,

$$\begin{aligned}\left\langle \tilde{\boldsymbol{v}}, \tilde{\boldsymbol{b}}\tilde{J} \right\rangle_{L^2(\tilde{\Omega})} &\leq \left\| \tilde{\boldsymbol{v}}\tilde{J}^{\frac{1}{2}} \right\|_{L^2(\tilde{\Omega})} \left\| \tilde{\boldsymbol{b}}\tilde{J}^{\frac{1}{2}} \right\|_{L^2(\tilde{\Omega})} \\ &\leq \frac{\nu C}{2} \left\| \tilde{\boldsymbol{v}}\tilde{J}^{\frac{1}{2}} \right\|^2_{L^2(\tilde{\Omega})} + \frac{1}{2\nu C} \left\| \tilde{\boldsymbol{b}}\tilde{J}^{\frac{1}{2}} \right\|^2_{L^2(\tilde{\Omega})} \quad \text{for} \quad \nu C > 0\end{aligned}$$

by Young's inequality. Similarly,

$$\left\langle \tilde{\boldsymbol{v}}, \tilde{\boldsymbol{P}}\tilde{\boldsymbol{N}} \right\rangle_{L^2(\tilde{\Gamma})} \leq \frac{\nu C}{2} ||\tilde{\boldsymbol{v}}||^2_{L^2(\tilde{\Gamma})} + \frac{1}{2\nu C} \left\| \tilde{\boldsymbol{P}}\tilde{\boldsymbol{N}} \right\|^2_{L^2(\tilde{\Gamma})} \quad \text{for} \quad \nu C > 0.$$

This done, the mathematical machinery necessary to the long–term stability analysis is in place.

THEOREM 2 (LONG–TERM STABILITY) *A sufficient condition for the completely general reference description to inherit the property of long–term stability*

$$\lim_{t\to\infty} \sup \tilde{K}(\tilde{\boldsymbol{v}}) \leq \frac{M^2}{2\nu^2 C^2}$$

under conditions of time–dependent loading (an intrinsic feature of real flows and the Navier–Stokes equations), where this time–dependent loading and the speed of the free surface is bounded in such a way that

$$\rho \left\| \tilde{\boldsymbol{b}}\tilde{J}^{\frac{1}{2}} \right\|^2_{L^2(\tilde{\Omega})} + \left\| \tilde{\boldsymbol{P}}\tilde{\boldsymbol{N}} \right\|^2_{L^2(\tilde{\Gamma})} + \nu^2 C^2 ||\tilde{\boldsymbol{v}}||^2_{L^2(\tilde{\Gamma})} \leq M^2,$$

is that the description becomes purely Lagrangian at free boundaries.

PROOF: Using Lemma 4 in equation (13), then applying the above bound,

$$\frac{d\tilde{K}(\tilde{\boldsymbol{v}})}{dt} + \nu C \tilde{K}(\tilde{\boldsymbol{v}}) \leq \frac{M^2}{2\nu C}.$$

Using the Gronwall lemma (see HIRSCH and SMALE [2]) leads to the differential inequality

$$\frac{d\tilde{K}(\tilde{\boldsymbol{v}})}{dt} \leq \frac{M^2}{2\nu C} e^{-\nu C t},$$

which, when solved, yields

$$\tilde{K}(\tilde{\boldsymbol{v}}) \leq e^{-\nu C t} \tilde{K}(\tilde{\boldsymbol{v}} \,|_{t=t_0}) + \left(1 - e^{-\nu C t}\right) \frac{M^2}{2\nu^2 C^2}.$$

This in turn implies

$$\lim_{t\to\infty} \sup \tilde{K}(\tilde{\boldsymbol{v}}) \leq \frac{M^2}{2\nu^2 C^2}.$$

The preceding analyses lead to natural notions of nonlinear dissipation in the absence of forcing and long–term stability under conditions of time–dependent loading for the analytic problem.

4 The Energetic Implications of the Time Discretisation

This section is concerned with establishing a class of time discretisations which inherit the self–same energetic properties (nonlinear dissipation in the absence of forcing and long–term stability under conditions of time dependent loading) as the analytic problem, irrespective of the time increment employed. In this section a generalised, Euler difference time–stepping scheme for the completely general reference equation is formulated and the energetic implications investigated in a similar vein as the analytic equations in the previous section.

This stability analysis is inspired by the approach of others to schemes for the conventional Navier–Stokes equations. The desirability of the attributes identified as key energetic properties is recognised and they have been used as a benchmark in the analysis of various of the conventional, Eulerian Navier–Stokes schemes by a host of authors. Related work on the conventional, Eulerian Navier–Stokes equations can be found in a variety of references, for example Temam [11] and Simo and Armero [8].

The analyses presented here are extended, not only in the sense that they deal with the completely general reference equation, but also in that non–zero boundaries, so–called free boundaries and time–dependent loads are able to be taken into account (the former two as a consequence of the new inequality on page 10). The findings of this work have profound consequences for the implementation of the deforming reference equations. It is significant that many algorithms used for long–term simulation do not automatically inherit the fundamental qualitative features of the dynamics.

A Generalised Time–Stepping Scheme

An expression for a generalised Euler difference time–stepping scheme can be formulated by introducing an "intermediate" velocity

$$\tilde{\boldsymbol{v}}_{n+\alpha} \equiv \alpha\ \tilde{\boldsymbol{v}}\ |_{t+\Delta t} + (1-\alpha)\ \tilde{\boldsymbol{v}}\ |_{t} \qquad \text{for} \qquad \alpha \in [0,1] \tag{14}$$

to the variational momentum equation (equation (9) on page 7) where $\tilde{\boldsymbol{v}}\ |_t$ and $\tilde{\boldsymbol{v}}\ |_{t+\Delta t}$ are the solutions at times t and $t+\Delta t$ respectively, Δt being the time step. It is in this way that a generalised time–discrete approximation of the momentum equation

$$\begin{aligned}
\frac{\rho}{\Delta t}\left\langle \tilde{\boldsymbol{w}}, (\tilde{\boldsymbol{v}}_{n+1} - \tilde{\boldsymbol{v}}_n)\tilde{J}_{n+\alpha}\right\rangle_{L^2(\tilde{\Omega}_{n+\alpha})} = \\
\left\langle \tilde{p}\tilde{\nabla}\tilde{\boldsymbol{w}}, \tilde{\boldsymbol{F}}^{-t}_{n+\alpha}\tilde{J}_{n+\alpha}\right\rangle_{L^2(\tilde{\Omega}_{n+\alpha})} - 2\mu\left\langle \tilde{\boldsymbol{D}}(\tilde{\boldsymbol{w}}), \tilde{\boldsymbol{D}}(\tilde{\boldsymbol{v}}_{n+\alpha})\tilde{J}_{n+\alpha}\right\rangle_{L^2(\tilde{\Omega}_{n+\alpha})} \\
-\rho\left\langle \tilde{\boldsymbol{w}}, (\tilde{\nabla}\tilde{\boldsymbol{v}}_{n+\alpha})\tilde{\boldsymbol{F}}^{-1}_{n+\alpha}\left(\tilde{\boldsymbol{v}}_{n+\alpha} - \tilde{\boldsymbol{v}}^{ref}_{n+\alpha}\right)\tilde{J}_{n+\alpha}\right\rangle_{L^2(\tilde{\Omega}_{n+\alpha})} \\
+\rho\left\langle \tilde{\boldsymbol{w}}, \tilde{\boldsymbol{b}}_{n+\alpha}\tilde{J}_{n+\alpha}\right\rangle_{L^2(\tilde{\Omega}_{n+\alpha})} + \left\langle \tilde{\boldsymbol{w}}, \tilde{\boldsymbol{P}}_{n+\alpha}\tilde{\boldsymbol{N}}_{n+\alpha}\right\rangle_{L^2(\tilde{\Gamma}_{n+\alpha})}
\end{aligned} \tag{15}$$

is derived, where $\langle\ .\ \rangle_{L^2(\tilde{\Omega}_{n+\alpha})}$ denotes the L^2 inner product over the deforming domain at time $t+\alpha\Delta t$. $\tilde{\Gamma}_{n+\alpha}$, $\tilde{\boldsymbol{F}}_{n+\alpha}$, $\tilde{J}_{n+\alpha}$, $\tilde{\boldsymbol{D}}_{n+\alpha}$, $\tilde{\boldsymbol{P}}_{n+\alpha}$, and $\tilde{\boldsymbol{b}}_{n+\alpha}$ are likewise defined to be the relevant quantities evaluated at time $t+\alpha\Delta t$.

It will presently become clear that it makes sense to perform the analyses for the time–discrete equation in the context of divergence free rates of reference deformation only. This is since relevant energy terms are not readilly recovered from the time-discrete equations for deforming references in general. This investigation is accordingly restricted to a subclass of reference deformations in which "reference volume" is conserved. This is for reasons of expedience alone and the subclass of deformations is thought to be representative.

ASSUMPTION 1 *The assumptions* $\tilde{J}_n = \tilde{J}_{n+\alpha}$ *and* $\tilde{J}_{n+1} = \tilde{J}_{n+\alpha}$ *are made so that*

$$\tilde{K}(\tilde{\boldsymbol{v}}_n) = \frac{1}{2}\rho \left|\left| \tilde{\boldsymbol{v}}_n \tilde{J}_n^{\frac{1}{2}} \right|\right|^2_{L^2(\tilde{\Omega}_n)} = \frac{1}{2}\rho \left|\left| \tilde{\boldsymbol{v}}_n \tilde{J}_{n+\alpha}^{\frac{1}{2}} \right|\right|^2_{L^2(\tilde{\Omega}_{n+\alpha})}$$

and

$$\tilde{K}(\tilde{\boldsymbol{v}}_{n+1}) = \frac{1}{2}\rho \left|\left| \tilde{\boldsymbol{v}}_{n+1} \tilde{J}_{n+1}^{\frac{1}{2}} \right|\right|^2_{L^2(\tilde{\Omega}_{n+1})} = \frac{1}{2}\rho \left|\left| \tilde{\boldsymbol{v}}_{n+1} \tilde{J}_{n+\alpha}^{\frac{1}{2}} \right|\right|^2_{L^2(\tilde{\Omega}_{n+\alpha})}$$

(by equation (14) and since the volume of material over which integration is being performed is constant).

REMARK: Notice that $\dfrac{\tilde{J}_{n+1} - \tilde{J}_n}{\Delta t} = \tilde{J} \operatorname{div} \boldsymbol{v}^{ref}_{n+\alpha}$, the discrete form of $\dfrac{\partial \tilde{J}}{\partial t} = \tilde{J} \operatorname{div} \boldsymbol{v}^{ref}$, can consequently be rewritten as

$$\operatorname{div} \boldsymbol{v}^{ref}_{n+\alpha} = 0$$

under the conditions of the above assumption. It is for the practical expedience afforded by Assumption 1 alone that this analysis is limited to instances in which $\operatorname{div} \boldsymbol{v}^{ref}_{n+\alpha} = 0$.

The following lemma will establish that the rate of energy change associated with the convective term vanishes as a result of the assumption.

LEMMA 5 (DISCRETE CONVECTIVE ENERGY RATE) *The discrete convective term*

$$-\rho \left\langle \tilde{\boldsymbol{w}}, (\tilde{\nabla}\tilde{\boldsymbol{v}}_{n+\alpha})\tilde{\boldsymbol{F}}^{-1}_{n+\alpha} \left(\tilde{\boldsymbol{v}}_{n+\alpha} - \tilde{\boldsymbol{v}}^{ref}_{n+\alpha} \right) \tilde{J}_{n+\alpha} \right\rangle_{L^2(\tilde{\Omega}_{n+\alpha})}$$

vanishes under circumstances of $\operatorname{div} \boldsymbol{v}^{ref}_{n+\alpha} = 0$ *and a purely Lagrangian description is used at free boundaries (alternatively boundaries are of the fixed, impermeable type).*

PROOF: The operator $\left\langle \cdot\, , (\tilde{\nabla}\, \cdot\,)\tilde{\boldsymbol{F}}^{-1}\tilde{\boldsymbol{w}}\tilde{J} \right\rangle_{L^2(\tilde{\Omega})}$ is skew–symmetric for

$$\tilde{\boldsymbol{w}} \in W = \left\{ \tilde{\boldsymbol{w}} \ : \ (\tilde{\nabla}\tilde{\boldsymbol{w}}) : \tilde{\boldsymbol{F}}^{-t} = 0 \ \text{ on } \tilde{\Omega};\ \tilde{\boldsymbol{w}} = \boldsymbol{0} \text{ or } \tilde{\boldsymbol{F}}^{-t}\tilde{\boldsymbol{N}} \cdot \tilde{\boldsymbol{w}} = 0 \text{ on } \tilde{\Gamma} \right\}$$

by equation (11) on page 13. In instances where a purely Lagrangian description is used to track free boundaries $\tilde{\boldsymbol{v}}_{n+\alpha} - \tilde{\boldsymbol{v}}^{ref}_{n+\alpha}$ vanishes. At fixed, impermeable boundaries $\tilde{\boldsymbol{F}}^{-t}_{n+\alpha}\tilde{\boldsymbol{N}}_{n+\alpha} \cdot \tilde{\boldsymbol{v}}_{n+\alpha}$ vanishes. The condition at the boundary is therefore satisfied, under

all of the afore–mentioned circumstances. Apply the stipulated condition $\tilde{\nabla}\tilde{\boldsymbol{v}}^{ref} : \tilde{\boldsymbol{F}}^{-t} = 0$ and set $\tilde{\boldsymbol{w}} = \tilde{\boldsymbol{v}}_{n+\alpha} - \tilde{\boldsymbol{v}}^{ref}_{n+\alpha}$ etc.

REMARK: Recall that in the investigation of the analytic problem, a term arising from the manipulation of the acceleration containing term (the term containing the rate of change of the Jacobian) cancelled with the convective energy. It is therefore not surprising that assumptions pertaining to the acceleration containing term (in particular to the rate of change of the Jacobian) in the discrete problem will, once made, also be necessary for the corresponding discrete convective energy term to vanish (reffering to the $\operatorname{div} \boldsymbol{v}^{ref} = 0$ condition of Lemma 5). This is a good prognosis for the energetic behaviour of the discrete problem in circumstances of reference deformations excluded by Assumption 1.

This concludes the preliminaries required for the analysis of the time–discrete equation.

4.1 Nonlinear Dissipation in the Absence of Forcing

The following analysis establishes a class of time–stepping schemes which exhibit nonlinear dissipation in the absence of forcing regardless of the time increment employed.

THEOREM 3 (NONLINEAR DISSIPATION IN THE ABSENCE OF FORCING) *Suppose that the description is pure Lagrangian at any free boundaries and that the deformation rate of the reference is divergence free. A sufficient condition for the kinetic energy associated with the generalised class of time–stepping schemes to decay nonlinearly*

$$\tilde{K}(\tilde{\boldsymbol{v}}_{n+1}) - \tilde{K}(\tilde{\boldsymbol{v}}_n) \;\leq\; -\Delta t\; 2\mu \left|\left| \tilde{\boldsymbol{D}}(\tilde{\boldsymbol{v}}_{n+\alpha}) \tilde{J}^{\frac{1}{2}}_{n+\alpha} \right|\right|^2_{L^2(\tilde{\Omega}_{n+\alpha})}$$

in the absence of forcing and irrespective of the time increment employed, is that the scheme is as, or more, implicit than central difference. That is

$$\alpha \geq \frac{1}{2}.$$

PROOF: Expressing the intermediate velocities $\tilde{\boldsymbol{v}}_{n+\frac{1}{2}}$ and $\tilde{\boldsymbol{v}}_{n+\alpha}$ in terms of equation (14) and subtracting, the result

$$\tilde{\boldsymbol{v}}_{n+\alpha} = \left(\alpha - \frac{1}{2}\right)(\tilde{\boldsymbol{v}}_{n+1} - \tilde{\boldsymbol{v}}_n) + \tilde{\boldsymbol{v}}_{n+\frac{1}{2}} \tag{16}$$

is obtained. The first step towards formulating an expression involving the kinetic energy of the generalised time stepping–scheme (15) is to replace the arbitrary vector, $\boldsymbol{w}$, with $\tilde{\boldsymbol{v}}_{n+\alpha}$. By further substituting (16) into (15) and eliminating the pressure containing term in a similar manner to that in Theorem 1, an expression involving the difference in kinetic energy over the duration of a single time step is obtained.

The vector $\tilde{\boldsymbol{v}} - \tilde{\boldsymbol{v}}^{ref}$ vanishes in instances where a purely Lagrangian description is used to track free boundaries. The quantity $\tilde{\boldsymbol{F}}^{-t}_{n+\alpha}\tilde{\boldsymbol{N}}_{n+\alpha} \cdot \tilde{\boldsymbol{v}}_{n+\alpha}$ vanishes where boundary conditions are of a fixed impermeable type. The condition at the boundary for Lemma 5 is

therefore satisfied. Incompressibility and a restriction on reference deformations to those for which $\mathrm{div}\boldsymbol{v}^{ref}_{n+\alpha}$ is zero ensure that the remaining Lemma 5 condition is satisfied.

The equation

$$\begin{aligned}\tilde{K}(\tilde{\boldsymbol{v}}_{n+1}) - \tilde{K}(\tilde{\boldsymbol{v}}_n) &= -\rho\left(\alpha - \frac{1}{2}\right)\left|\left|(\tilde{\boldsymbol{v}}_{n+1} - \tilde{\boldsymbol{v}}_n)\,\tilde{J}^{\frac{1}{2}}_{n+\alpha}\right|\right|^2_{L^2(\tilde{\Omega}_{n+\alpha})} \\ &\quad -\Delta t\; 2\mu\left|\left|\tilde{\boldsymbol{D}}(\tilde{\boldsymbol{v}}_{n+\alpha})\tilde{J}^{\frac{1}{2}}_{n+\alpha}\right|\right|^2_{L^2(\tilde{\Omega}_{n+\alpha})} + \Delta t\rho\left\langle\tilde{\boldsymbol{v}}_{n+\alpha}, \tilde{\boldsymbol{b}}_{n+\alpha}\tilde{J}_{n+\alpha}\right\rangle_{L^2(\tilde{\Omega}_{n+\alpha})} \\ &\quad +\Delta t\left\langle\tilde{\boldsymbol{v}}_{n+\alpha}, \tilde{\boldsymbol{P}}_{n+\alpha}\tilde{\boldsymbol{N}}_{n+\alpha}\right\rangle_{L^2(\tilde{\Gamma}_{n+\alpha})}, \end{aligned} \tag{17}$$

is then obtained. Since it is assumed that there is no forcing,

$$\begin{aligned}\tilde{K}(\tilde{\boldsymbol{v}}_{n+1}) - \tilde{K}(\tilde{\boldsymbol{v}}_n) &\leq -\rho\left(\alpha - \frac{1}{2}\right)\left|\left|(\tilde{\boldsymbol{v}}_{n+1} - \tilde{\boldsymbol{v}}_n)\,\tilde{J}^{\frac{1}{2}}_{n+\alpha}\right|\right|^2_{L^2(\tilde{\Omega}_{n+\alpha})} \\ &\quad -\Delta t\; 2\mu\left|\left|\tilde{\boldsymbol{D}}(\tilde{\boldsymbol{v}}_{n+\alpha})\tilde{J}^{\frac{1}{2}}_{n+\alpha}\right|\right|^2_{L^2(\tilde{\Omega}_{n+\alpha})}.\end{aligned}$$

Thus the kinetic energy inherent to the algorithmic flow decreases nonlinearly in the absence of forcing, irrespective of the time increment employed and for arbitrary initial conditions provided that

$$\alpha \geq \frac{1}{2} \qquad \text{and} \qquad \mathrm{div}\,\boldsymbol{v}^{ref}_{n+\alpha} = 0.$$

The former requirement translates directly into one specifying the use of schemes as, or more, implicit than central difference. Only for descriptions which become fully Lagrangian at free boundaries can it be guaranteed that energy will not be artificially introduced by way of the reference.

Remark: Notice (by Lemma 1) that for $\alpha = \frac{1}{2}$ an identical rate of energy decay

$$\frac{\tilde{K}(\tilde{\boldsymbol{v}}_{n+1}) - \tilde{K}(\tilde{\boldsymbol{v}}_n)}{\Delta t} \leq -2\nu C\tilde{K}(\tilde{\boldsymbol{v}}_{n+\alpha})$$

is obtained for the discrete approximation as was obtained for the equations.

4.2 Long–Term Stability under Conditions of Time–Dependent Loading

This second part of the time–discrete analysis establishes a class of time stepping schemes which exhibit long–term stability under conditions of time dependent loading irrespective of the time increment employed. The following lemma is necessary to the analysis and is concerned with devising a bound for the energy at an intermediate point in terms of energy values at either end of the time step.

Lemma 6 (Intermediate Point Energy) *The following bound applies*

$$\tilde{K}(\tilde{\boldsymbol{v}}_{n+\alpha}) \geq \alpha\,(\alpha - c + \alpha c)\,\tilde{K}(\tilde{\boldsymbol{v}}_{n+1}) + (1-\alpha)\left(1 - \alpha - \frac{\alpha}{c}\right)\tilde{K}(\tilde{\boldsymbol{v}}_n)$$

where c is some constant, $c > 0$.

PROOF: By Young's inequality

$$\left\|\tilde{\boldsymbol{v}}_{n+1}\tilde{J}^{\frac{1}{2}}_{n+\alpha}\right\|_{L^2(\tilde{\Omega}_{n+\alpha})}\left\|\tilde{\boldsymbol{v}}_{n}\tilde{J}^{\frac{1}{2}}_{n+\alpha}\right\|_{L^2(\tilde{\Omega}_{n+\alpha})} \leq \left(\frac{c}{2}\right)\left\|\tilde{\boldsymbol{v}}_{n+1}\tilde{J}^{\frac{1}{2}}_{n+\alpha}\right\|^2_{L^2(\tilde{\Omega}_{n+\alpha})} + \left(\frac{1}{2c}\right)\left\|\tilde{\boldsymbol{v}}_{n}\tilde{J}^{\frac{1}{2}}_{n+\alpha}\right\|^2_{L^2(\tilde{\Omega}_{n+\alpha})} \tag{18}$$

for $c > 0$. Writing $\tilde{K}(\tilde{\boldsymbol{v}}_{n+\alpha})$ explicitly in terms of the "intermediate" velocity definition, (14), leads to

$$\begin{aligned}
\tilde{K}(\tilde{\boldsymbol{v}}_{n+\alpha}) &= \alpha^2\tilde{K}(\tilde{\boldsymbol{v}}_{n+1}) + (1-\alpha)^2\tilde{K}(\tilde{\boldsymbol{v}}_{n}) + 2\alpha(1-\alpha)\left\langle\tilde{\boldsymbol{v}}_{n+1},\tilde{\boldsymbol{v}}_{n}\tilde{J}_{n+\alpha}\right\rangle_{L^2(\tilde{\Omega}_{n+\alpha})} \\
&\geq \alpha^2\tilde{K}(\tilde{\boldsymbol{v}}_{n+1}) + (1-\alpha)^2\tilde{K}(\tilde{\boldsymbol{v}}_{n}) \\
&\quad -2\alpha(1-\alpha)\left\|\tilde{\boldsymbol{v}}_{n+1}\tilde{J}^{\frac{1}{2}}_{n+\alpha}\right\|_{L^2(\tilde{\Omega}_{n+\alpha})}\left\|\tilde{\boldsymbol{v}}_{n}\tilde{J}^{\frac{1}{2}}_{n+\alpha}\right\|_{L^2(\tilde{\Omega}_{n+\alpha})} \\
&\geq \alpha\left[\alpha-(1-\alpha)c\right]\tilde{K}(\tilde{\boldsymbol{v}}_{n+1}) + (1-\alpha)\left[(1-\alpha)-\frac{\alpha}{c}\right]\tilde{K}(\tilde{\boldsymbol{v}}_{n})
\end{aligned}$$

using equation (18). The optimal choice of the constant c is established farther on.

The following theorem establishes a class of time–stepping schemes which exhibit long–term stability under conditions of time–dependent loading regardless of the time increment employed.

THEOREM 4 (LONG–TERM STABILITY) *Suppose that the description is pure Lagrangian at any free boundaries and that the rate at which the reference is deformed is divergence free. A sufficient condition for the algorithmic flow to exhibit long–term stability under conditions of time–dependent loading (intrinsic to real flows and the Navier–Stokes equations), assuming this time–dependent loading and the speed of the free surface is bounded in such a way that*

$$\rho\left\|\tilde{\boldsymbol{b}}_{n+\alpha}\tilde{J}^{\frac{1}{2}}_{n+\alpha}\right\|^2_{L^2(\tilde{\Omega}_{n+\alpha})} + \left\|\tilde{\boldsymbol{P}}_{n+\alpha}\tilde{\boldsymbol{N}}_{n+\alpha}\right\|^2_{L^2(\tilde{\Gamma}_{n+\alpha})} + \nu^2C^2\left|\left|\tilde{\boldsymbol{v}}_{n+\alpha}\right|\right|^2_{L^2(\tilde{\Gamma}_{n+\alpha})} \leq M^2,$$

is

$$\alpha > \frac{1}{2}.$$

PROOF: Substituting Lemma 4 (page 15) and Lemma 1 (page 12) into equation (17), applying the above bound and choosing $\alpha \geq \frac{1}{2}$ one obtains

$$\frac{\tilde{K}(\tilde{\boldsymbol{v}}_{n+1}) - \tilde{K}(\tilde{\boldsymbol{v}}_{n})}{\Delta t} + \nu C\tilde{K}(\tilde{\boldsymbol{v}}_{n+\alpha}) \leq \frac{M^2}{2\nu C}.$$

From this point on the argument used is identical to that of SIMO and ARMERO [8] for the conventional, Eulerian Navier–Stokes equations. Substitution of Lemma 6 leads to a recurrence relation,

$$\tilde{K}(\tilde{\boldsymbol{v}}_{n+1}) \leq \frac{1-\nu C(1-\alpha)(1-\alpha-\frac{\alpha}{c})\Delta t}{1+\nu C\alpha(\alpha-c+\alpha c)\Delta t}\tilde{K}(\tilde{\boldsymbol{v}}_{n}) + \frac{M^2\Delta t}{2\nu C\left[1+\nu C\alpha(\alpha-1+\alpha c)\Delta t\right]}.$$

Using this recurrence relation to take cognisance of the energy over all time steps,

$$\begin{aligned}\tilde{K}(\tilde{\boldsymbol{v}}_{n+1}) &\leq \left[\frac{1-\nu C(1-\alpha)(1-\alpha-\frac{\alpha}{c})\Delta t}{1+\nu C\alpha(\alpha-c+\alpha c)\Delta t}\right]^n \tilde{K}(\tilde{\boldsymbol{v}}_0) \\ &\quad + \frac{M^2\Delta t}{2\nu C\left[1+\nu C\alpha(\alpha-c+\alpha c)\Delta t\right]} \sum_{k=0}^{n-1}\left[\frac{(1-\nu C(1-\alpha)(1-\alpha-\frac{\alpha}{c})\Delta t)}{1+\nu C\alpha(\alpha-c+\alpha c)\Delta t}\right]^k \end{aligned} \tag{19}$$

is obtained. An infinite geometric series which converges so that

$$\begin{aligned}\lim_{n\to\infty}\sup \tilde{K}(\tilde{\boldsymbol{v}}_{n+1}) &\leq \frac{M^2\Delta t}{2\nu C\left[1+\nu C\alpha(\alpha-c+\alpha c)\Delta t\right]}\left[1\right. \\ &\qquad \left. - \frac{(1-\nu C(1-\alpha)(1-\alpha-\frac{\alpha}{c})\Delta t)}{1+\nu C\alpha(\alpha-c+\alpha c)\Delta t}\right]^{-1} \\ &= \frac{M^2}{2\nu C\left[\nu C\alpha(\alpha-c+\alpha c)+\nu C(1-\alpha)(1-\alpha-\frac{\alpha}{c})\right]}\end{aligned}$$

results, providing the absolute ratio of the series is less than unity. That is

$$\left|\frac{1-\nu C(1-\alpha)(1-\alpha-\frac{\alpha}{c})\Delta t}{1+\nu C\alpha(\alpha-c+\alpha c)\Delta t}\right| < 1.$$

Therefore either

$$-1-\nu C\alpha(\alpha-c+\alpha c)\Delta t < 1-\nu C(1-\alpha)\left(1-\alpha-\frac{\alpha}{c}\right)\Delta t$$

or

$$1-\nu C(1-\alpha)\left(1-\alpha-\frac{\alpha}{c}\right)\Delta t < 1+\nu C\alpha(\alpha-c+\alpha c)\Delta t \tag{20}$$

in order for the bound to exist. Notice, furthermore, that for this desired convergence to be unconditional (regardless of the time increment employed) requires

$$\alpha-c+\alpha c \geq 0. \tag{21}$$

The denominator in the series ratio might otherwise vanish for some value of Δt.

For $\alpha \in \left[\frac{1}{2}, 1\right]$ equation (20) and equation (21) together imply

$$\frac{(1-\alpha)}{\alpha} < c \leq \frac{\alpha}{(1-\alpha)}$$

which in its turn implies

$$\frac{(1-\alpha)}{\alpha} < \frac{\alpha}{(1-\alpha)}.$$

The choice of the parameter $\alpha > \frac{1}{2}$ therefore leads to an infinite geometric series which forms the desired upper bound. The minimum value of this bound occurs for c chosen according to

$$\inf_{\frac{(1-\alpha)}{\alpha}<c\leq\frac{\alpha}{(1-\alpha)}} \frac{1}{\nu C\alpha(\alpha - c + \alpha c) + \nu C(1-\alpha)(1-\alpha-\frac{\alpha}{c})} \quad = \quad \frac{1}{\nu C(2\alpha-1)^2}.$$

The value of this upper bound, which occurs for the choice of the parameter $\alpha > \frac{1}{2}$, is then

$$\lim_{n\to\infty} \sup \tilde{K}(\tilde{\boldsymbol{v}}_{n+1}) \quad \leq \quad \frac{M^2}{2\nu^2 C^2(2\alpha-1)^2}.$$

In this way one arrives at a class of algorithms which are unconditionally (irrespective of the time increment employed) stable.

REMARK: Notice that for $\alpha = 1$ one obtains an identical energy bound for the discrete approximation as was obtained for the equations.

5 An "Updated" Approach and a Simplified Implementation

Ever burgeoning deformation gradients accumulate for a straight forward implementation of the equations. Using an "updated" approach is one way of coping with this otherwise rather daunting prospect. An "updated" approach is the result of a little, well–worthwhile lateral thinking. An "updated" approach amounts to choosing a new referential configuration during each time step.

In the case of time stepping schemes based about a single instant (eg. the generalised class of Euler difference schemes investigated in Section 4) a considerably simplified implementation can further be achieved by a particularly appropriate choice of configurations. Making the choice of a referential configuration which coincides with the spatial configuration at the instant about which the time stepping scheme is based allows the deformation gradient to be omitted altogether (the deformation gradient is identity under such circumstances). For such implementations (those which require evaluation about a single point only) no error arises from the use of the equations cited in HUGHES, LIU and ZIMMERMAN [4],

$$\rho\left(\frac{\partial \boldsymbol{v}}{\partial t} + \nabla \boldsymbol{v}(\boldsymbol{v} - \boldsymbol{v}^{ref})\right) \quad = \quad \rho\boldsymbol{b} + \operatorname{div}\boldsymbol{\sigma} \tag{22}$$

$$\operatorname{div}\boldsymbol{v} \quad = \quad 0. \tag{23}$$

These equations are not valid for any, arbitrary choice of reference or if the implementation requires the equation to be evaluated at more than one point within each time step (eg. a Runge–Kutta or finite–element–in–time scheme). It is important to remember that in a discrete context the reference configuration is fixed for the duration of the entire

time increment. Although the referential configuration is hypothetical and can be chosen arbitrarily for each time step, once chosen it is static for the duration of the entire time step. Once the coincidence of configurations is ordained at a given instant, $\tilde{\boldsymbol{F}}$ is defined by the deformation, both before and after, and must be consistant.

There would seem to be no reason why one would wish to define the deformation about a configuration other than that at the instant about which the implementation is based (assuming the implementation used is indeed based about a single point eg. a finite difference) thereby involving deformation gradients. Resolving the resulting difficulties associated with the deformation gradients by means of a perturbation seems unnecessarily complicated in the light of the above reasoning.

6 Conclusions

The correct equations, which describe the motion of an incompressible, Newtonian fluid and which are valid for a completely general range of reference deformations, are equations (7) and (8). For implementations requiring the equations to be evaluated about a single instant within each time step only (eg. finite differences), the deformation gradients may be assumed identity i.e. the equations of Hughes, Liu and Zimmerman [4] (equations (22) and (23)) will suffice.

In this work it is shown (as was hoped) that nonlinear, exponential–type dissipation in the absence of forcing and long–term stability under conditions of time dependent loading are properties automatically inherited by deforming reference descriptions. The single provisor is that such descriptions become fully Lagrangian at any moving boundaries. These properties are intrinsic to real flows and the conventional, Eulerian Navier–Stokes equations.

Relevant energy terms are not readily recovered from the time–discrete equations for deforming references in general. Only for divergence free rates of reference deformation which become fully Lagrangian at free boundaries could it consequently be guaranteed that energy would not be artificially introduced to the algorithmic flow by way of the reference. The divergence free assumption was made for reasons of expedience alone and the limitations of the time–discrete analysis are consequently not expected to detract from the use of the method in any way. This is especially so when it is considered that, a term arising from the manipulation of the acceleration containing term (the term containing the rate of change of the Jacobian) cancelled with the convective energy in the investigation of the analytic problem and that assumptions pertaining to the acceleration containing term (in particular to the rate of change of the Jacobian) in the discrete problem were, once made, also necessary for the corresponding discrete convective energy term to vanish (reffering to the $\operatorname{div} \boldsymbol{v}^{ref} = 0$ condition of Lemma 5). If one were to be overly cautious on this basis one would be faced with the additional challenge of enforcing a fully Lagrangian description for nodes situated on any free boundaries, while deforming elements would be required to deform at a rate which is divergence free. Such a totally divergence free description may, however, not be possible. An alternative strategy would be to use a fully Lagrangian description. Both the purely Lagrangian and purely Eulerian fluid

descriptions have divergence free rates of distortion.

There are inherent problems with using certain classes of time–stepping schemes and the use of finite difference schemes more implicit than central difference is consequently advocated. Such differences exhibit the key energetic properties (nonlinear, exponential–type dissipation in the absence of forcing and long–term stability under conditions of time dependent loading) irrespective of the time increment employed. A backward difference is the obvious choice. Calculations at time $t + \alpha\Delta t$ would require an intermediate mesh and associated quantities for instances in which $\alpha \neq 1$ (since $\alpha > \frac{1}{2}$).

The author recommends a strategy in which a predominantly Eulerian description is used, where possible, for the bulk of the problem (from an efficiency point of view) and the completely general reference description for the remainder is appropriate. Purely Eulerian descriptions have the advantage of a "one off" finite element construction and involve none of the hazards of a badly distorted reference.

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